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Comparison of relativistic partial wave calculations of triply differential electron-atom bremsstrahlung with simpler theories

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Abstract

The results of our exact partial wave triply differential electron–atom bremsstrahlung calculations are compared with those of simpler theories. This triply differential cross section is expected to provide a more sensitive test of the adequacy of theoretical descriptions of the process than the doubly differential cross section, not observing the scattered electron. Our method is an extension and reformulation of the relativistic partial wave and multipole decomposition method of Tseng and Pratt, which was only used to calculate the doubly differential cross section. We compare our calculations to those using the Bethe-Heitler equation (first Born approximation including form factors) and to calculations using the formalism of Elwert and Haug (which utilize approximate Coulomb wave functions and form factors). We examine silver and heavier elements, where the validity of the simpler theories is in question. For such elements we find no significant region of agreement with the Bethe-Heitler theory. We find some agreement with the theory of Elwert and Haug at small momentum transfers, improving with increasing energy. We find poor agreement of our results with the Elwert-Haug results for large momentum transfers. The differences from simpler theories are indeed much greater in the triply differential cross section than in the double differential

cross section.

I. INTRODUCTION

Bremsstrahlung, the emission of a photon by an electron scattering from an atom, is a fundamental test process in which to study the interaction of a photon and an electron in the field of an atom. While the final state in this process includes both the scattered electron and the emitted photon, and is described by a triply differential cross section $d^3\sigma$, the bulk of experimental and theoretical efforts in studying electron-atom bremsstrahlung has excluded observation of the outgoing electron. As such, the results of these efforts are a doubly differential cross section $d^2\sigma$, averaged over observable parameters of the outgoing electron (spin and momentum). However, there has been some experimental and theoretical work which includes observation of the scattered electron in coincidence with the emitted photon. It is often argued that such more complete experiments provide a more stringent test of the theoretical models [1]. Here we present a comparison of predictions from our new calculations, which use a full relativistic partial wave and multipole decomposition, retaining all important multipoles and partial waves, and the simpler approximations of Bethe and Heitler [2] (lowest order Born approximation) and Elwert and Haug [3] (using Sommerfeld-Maue wave functions). We indeed find far larger differences from the simpler theories for $d^3\sigma$ than for $d^2\sigma$.

In a previous paper [4] we presented first results from our triply differential electron–atom cross section (and polarization correlation) code, comparing with available experimental data (all for coplanar geometries). We also compared the predictions of the simpler theories for these cases. The current work extends the numerical comparison of the theories to a larger range of parameters, including situations where no experimental data is presently available. We examine both coplanar and non-coplanar geometries over a broad range of scattering angles and electron and photon energies. As with the previous work, we focus our discussion on heavy elements (atomic number $Z \geq 47$), where the validity of both of the simpler theories is in doubt. (Verification of our code was achieved, in part, through comparisons with the simpler theories for lighter elements [4]).

In Section II we discuss these various theories of bremsstrahlung. In Section III we present results from our numerical comparisons and, in Section IV, we summarize our conclusions and suggest future experimental and theoretical work.

II. DISCUSSION OF THEORIES

All of the theories discussed here are theories of bremsstrahlung accompanying electron scattering from a static central potential. That is, they are all theories within the independent particle approximation (IPA). We will be comparing "exact" predictions of such a theory (in a partial wave and multipole decomposition numerically summed to convergence) with simpler approximations to such a theory. We will identify circumstances in which the more exact treatment matters. In this work we will only consider scattering from neutral atoms (although each of the theories may also be applied to ionic species). Within the IPA, except for the screening of the nuclear charge, effects of the bound atomic electrons are ignored. The potentials used here are the relativistic self-consistent field (Dirac-Slater) potentials discussed by Liberman [5], but without a Latter tail [6].

We utilize a coordinate system in which the photon momentum $\vec{k}$ defines the z axis, and the $x-z$ plane contains the initial electron momentum $\vec{p}_1$ and $\vec{k}$. For convenience we define θ_1 as the angle between $\vec{k}$ and the initial electron momentum $\vec{p}_1$, θ_2 as the angle between $\vec{k}$ and the final electron momentum $\vec{p}_2$, and ϕ_2 as the angle between the plane containing $\vec{k}$ and $\vec{p}_1$ and the plane containing $\vec{k}$ and $\vec{p}_2$. This coordinate system is shown in Figure 1. We define the energies of the initial and final electron and the photon as E_1, E_2 and k, respectively. We also define the momentum transfered to the nucleus $\vec{q} = \vec{p}_1 - \vec{p}_2 - \vec{k}$. The quantity of interest in our discussion is the triply differential cross section $d^3\sigma \equiv \frac{d^3\sigma}{dk\, d\Omega_k\, d\Omega_2}$ for radiation in scattering of an unpolarized electron from a neutral atom of atomic number Z, where the polarization properties of the radiated photon and scattering electron are not observed. Here $d\Omega_k$ ($d\Omega_2$) is the finite solid angle subtended by the photon momentum (outgoing electron momentum). (Some discussion of the polarization correlations associated with the triply differential cross section was given in [4].)

A. Bethe-Heitler approximation

The Bethe-Heitler approximation [2] is the simplest of the three theories discussed here. It is also the most widely used, since the cross sections $d^3\sigma$, $d^2\sigma$ and $d\sigma$ for radiation in scattering from a Coulomb point charge can be expressed in closed form in terms of elementary functions. The Bethe-Heitler formula results from the application of the Born approximation for the interaction of the scattering electron with the potential, retaining only the first non-zero term. Detailed discussions of the Bethe-Heitler equation for $d^3\sigma$ can be found in [2] and a number of texts (see [7], for example). Note that, since it is derived from a first order perturbation for scattering in the potential, the Z dependence of the Bethe-Heitler result factors; $d^3\sigma/Z^2$ is Z independent. When scattering from a neutral or partially ionized atom, the Bethe-Heitler result includes multiplication of the matrix element by the form factor (as follows from the first Born treatment), introducing a more complicated Z dependence. The form factor [2]

$$F(q) = 1 - \frac{1}{Z}\int_0^\infty \rho(\vec{r})e^{i\vec{q}\cdot\vec{r}}d^3r, \tag{1}$$

where

$$\int_0^\infty \rho(\vec{r})d^3r = Z,$$

$\rho(\vec{r})$ is the charge density and $q = |\vec{q}|$ is the magnitude of the momentum transfer. In our calculations of the predictions of this theory form factors are obtained by numerically integrating Equation (1), using the charge density obtained from bound state wavefunctions in the Dirac-Slater potential discussed earlier, but with a Latter tail. Corrections to the Bethe-Heitler cross sections are of relative order $(Z\alpha/\beta)$, where β is the initial or final electron velocity, so the condition for the validity of this approximation is frequently written $Z\alpha/\beta \ll 1$. This condition must be satisfied for both the initial and final electron.

One of the most significant problems with the Bethe-Heitler result is that it predicts a zero cross section in the limit that the photon takes all of the initial electron's energy (the "tip" region of the spectrum). Approaching this limit, the outgoing electron has very little energy, and the $Z\alpha/\beta \ll 1$ condition for validity of the Born approximation is violated. The behavior in this limit can be fairly well corrected, at least for the spectrum $d\sigma$, by multiplying the Bethe-Heitler result by the Elwert factor [8]

$$f_E = \frac{\nu_f}{\nu_i} \frac{1 - e^{-2\pi\nu_i}}{1 - e^{-2\pi\nu_f}}, \tag{2}$$

where

$$\nu_{i,f} = Z\alpha/\beta_{i,f},$$

and $\beta_{i,f}$ are the initial and final velocity of the electron in units of the speed of light in vacuum. While originally derived in a comparison of non-relativistic Born and exact Coulomb results, the form (2) is usually taken as the relativistic generalization. The relativistic Elwert factor is simply the square of the ratio of the normalizations $(N_2/N_1)^2$ of the Sommerfeld-Maue wavefunctions. Multiplying the Bethe-Heitler cross section by the Elwert factor gives a nonzero cross section in the tip (hard photon) region and has no effect in the soft photon limit. A more detailed discussion of the Elwert factor can be found in [3].

B. The Elwert-Haug approximation

In the Elwert-Haug formulation, the interaction of the scattering electron with the static Coulomb field of the nucleus is included in the unperturbed Hamiltonian. However the exact wave functions in the Coulomb field are replaced by the approximate wavefunctions of Sommerfeld and Maue [9]. Bethe and Maximon [10] showed that these wavefunctions correctly represent all partial waves with $\kappa^2 \gg (Z\alpha)^2$ in the exact (Darwin) solution. Using the Sommerfeld-Maue wavefunctions and making some further approximation, Elwert and Haug obtained an analytic but rather complicated expression for the triply differential cross section, involving hypergeometric functions, as a function of Z, the momentum vectors and energies. (In a recent paper [11] Haug has also derived an expression for the polarization correlation C_{200}). We have found it more convenient in our calculations of the predictions of this theory to work from Elwert and Haug's results for the *matrix element*, computing the hypergeometric functions numerically. The Elwert-Haug result is expected to be valid in the point Coulomb potential for $\alpha Z \ll 1$ at all energies, or for all αZ at small scattering angles and sufficiently high incident and outgoing electron energy $E \gg mc^2$, and in suitable intermediate situations. In the case of the neutral atom, screening by the bound electrons is treated by taking the product of the form factor $F(q)$, calculated as above, with the Elwert-Haug result for the matrix element, thus making an additional approximation reducing the region of validity.

C. The exact partial wave method

In the exact partial wave method we include the exact numerical atomic potential in the unperturbed Hamiltonian. In this case the interaction (for bremsstrahlung from a neutral atom) is not with the Coulomb potential (as in the Elwert-Haug case) but with the screened Coulomb potential, so no form factor approximation is needed. Furthermore, no approximation in $(\kappa/Z\alpha)^2$ is made, but rather the full partial wave solutions are obtained numerically to an order in κ needed for convergence of the matrix element. The Dirac equation is separated into angular and (coupled) radial equations. We obtain numerical solutions of specified energy E of the radial Dirac equation, labeled by the quantum numbers κ and m. The quantum number $\kappa = \pm 1, \pm 2, \pm 3 \ldots$, corresponds to eigenvalues of the operator $-(\vec{\sigma} \cdot \vec{L} + 1)$; m is the eigenvalue of J_z, where $\vec{L}$ is the orbital angular momentum operator, and J_z is the z component of the total angular momentum operator. A linear combination of eigenstates $\psi_{\kappa m}$ is taken which obeys the appropriate boundary conditions (plane wave plus incoming or outgoing spherical wave). The photon interaction "operator", $\vec{\alpha} \cdot \vec{\epsilon}^{*} e^{-i\vec{k}\cdot\vec{r}}$, is expanded into a multipole series [12]. We may represent the three fold summation over incident and outgoing electron partial waves and photon angular momenta in the matrix element symbolically as $\sum_{\mathcal{K}} \mathcal{M}_{\mathcal{K}}$, where $\mathcal{K}(L, \kappa_1, \kappa_2)$ represents the photon angular momenta and parity and the electron angular momentum quantum numbers. Thus we write

$$M_{fi} = \sum_{\mathcal{K}} \mathcal{M}_{\mathcal{K}}, \tag{3}$$

and the expression for the polarized cross section is

$$d^3\sigma^{\mathrm{pol}} \propto \left| \sum_{\mathcal{K}} \mathcal{M}_{\mathcal{K}} \right|^2 . \tag{4}$$

The matrix elements in this summation involve radial integrals of the form

$$\begin{aligned} S_1 &= \int_0^\infty g_{\kappa_1} j_l(kr) f_{\kappa_2}\, dr, \\ S_2 &= \int_0^\infty g_{\kappa_2} j_l(kr) f_{\kappa_1}\, dr, \end{aligned}$$

where g_κ and f_κ are the small and large components of the initial and final spinors and $j_l(kr)$ are the spherical Bessel functions. Such integrals over rapidly oscillating functions are evaluated using the well tested computer program of Tseng [13]. These integrals depend only on p_1, p_2, k and Z. The summation in Eq. (3) to obtain the total matrix element is performed numerically before squaring to obtain the cross section. A more detailed description of our formalism can be found in [4].

III. RESULTS

We summarize our results in Table I. While a fully comprehensive comparison of these theories has not been undertaken, we have a significant amount of cross section data, which we believe is representative, over a broad range of parameters, for bremsstrahlung accompanying scattering from targets with $47 \leq Z \leq 79$. We have limited our calculations to energies considered in existing or anticipated experiments ($E_1 = 50$–450 keV, $0.4 < k/T_1 < 0.9$ where

T_1 is the electron's initial kinetic energy). We have calculated cross section data on a discrete grid over a full 4π solid angle for many energies and several Z within these ranges. We have performed calculations in the non-coplanar geometry, as well as in the coplanar geometry, in anticipation of experimental data [14,15]. Here we compare our results with those of the simpler theories for these ranges of parameters. Comparisons of these theories have been previously made for the case of the doubly differential cross section [16,17]. We emphasize here when comparisons of the triply differential cross section contrast with, or agree with, the conclusions from the comparisons of $d^2\sigma$. In all cases, our integrated cross section data agree with reported $d^2\sigma$ data where it exists, and we do not reach any conclusions for $d^2\sigma$ which contradict those of Tseng and Pratt [16] or Fink [17].

We find, however, that for $d^3\sigma$, in general, the Bethe-Heitler results and the Elwert-Bethe-Heitler results are frequently more than a factor of two too small or too large in comparison to the exact partial wave calculations. For these higher Z elements no substantial region of systematic agreement with either version of this theory was found in this work, though one can see trends of improving agreement as Z decreases. For these energies we find that the Elwert-Haug predictions are significantly closer to ours than are Bethe-Heitler predictions, even for $Z = 79$, for small momentum transfers. However, for large momentum transfers, or for large emission angles, $\theta_1 > 45°$, we find that the Elwert-Haug results approach the Elwert-Bethe-Heitler results and, correspondingly, significantly underestimate our results.

We may summarize the results of Fink [18] and Tseng and Pratt [16] on the agreement of the simpler theories with the exact partial wave treatment for $d^2\sigma$ [17] for heavier elements ($Z > 60$) at energies similar to those considered here: 1) Bethe-Heitler results away from the tip show differences (10-50%) from exact partial wave results, typically underestimating. 2) Use of the Elwert factor typically improves the Bethe-Heitler result. 3) For this energy range (though not at higher energy) Elwert-Haug results underestimate the exact partial wave results, deviating as significantly as Elwert-Bethe-Heitler results. For $d^2\sigma$ the Elwert-Haug result was generally no improvement on Bethe-Heitler. See the schematic summary in Table I. A somewhat more detailed discussion of the agreement between the theories can be found in [17,13].

In contrast to these results for $d^2\sigma$, we find (as already indicated and as we describe in more detail below) the following results for $d^3\sigma$: 1) We find more extreme differences than in $d^2\sigma$ between the Bethe-Heitler and exact partial wave results (100-400%), and we find that the sign of the difference between the two theories changes as a function of the momentum transfer q. 2) We find that for these energies use of the Elwert factor as defined in Eq. (2) is inappropriate at small momentum transfers. 3) We find some region of agreement with the Elwert-Haug theory at small momentum transfers, particularly for high incident and outgoing electron energies, but not for large momentum transfers. In the following subsections we show representative data and compare these theories in more detail. See also the schematic summary in Table I.

A. Comparison with (Elwert-)Bethe-Heitler approximation

In Figure 2 we show predictions of the various theories for the reduced cross section $d^3\sigma/Z^2$, as a function of θ_2, for the simple *forward photon emission* case of $\theta_1 = 0°$. The case shown is for Gold ($Z = 79$) at $E_1 = 300$ keV and $E_2 = 150$ keV, an example of equal energy sharing between the photon and outgoing electron. The data was smoothed using a cubic spline to increase legibility. Due to the excessive amount of computation time involved, our calculations were made on a grid at 9° increments (starting from $\theta_2 = 0°$), with additional grid points in the peak region of the cross section. We see that the Bethe-Heitler and Elwert-Bethe-Heitler results significantly overestimate the cross section near the peak. Also, the peak of the Bethe-Heitler cross section is narrower than that in the exact partial wave result: at larger angles the Bethe-Heitler result is underestimating the cross section. Inclusion of the multiplicative Elwert factor somewhat improves the large angle behavior but worsens the results near the peak in the cross section. In this simple case, where the incident electron and outgoing photon momenta are collinear, the cross section is a single-valued function of momentum transfer. In Figure 2 we have indicated the momentum transfers as well as the scattering angles. The kinematically allowed momentum transfer range in this case is between $q_{\min} \approx 0.1\,mc$ and $q_{\max} \approx 1.8\,mc$ (m is the mass of the electron and c the velocity of light in vacuum), with the limits corresponding to the outgoing electron parallel or anti-parallel to the incident electron. We see that at low momentum transfers ($q < 0.35\,mc$) the Bethe-Heitler cross sections are systematically larger than the exact partial wave results, even larger if using the Elwert factor. At larger momentum transfers the Bethe-Heitler results are systematically smaller than the exact partial wave results; use of the Elwert factor give some improvement.

In Figure 3 we show the corresponding forward emission cross section data for $Z = 47$, $E_1 = 100$ keV, $E_2 = 50$ keV. We see that for this lower Z the exact partial wave results are beginning to peak more sharply, but the Bethe-Heitler results still significantly underestimate the cross sections at large scattering angles and large momentum transfers. The broadening of the peak for increasing Z in the exact partial wave calculations was also seen in the $d^2\sigma$ results of Fink [18] and Tseng [16]. While we are showing a lower energy case than previously, the results are similar throughout the energy range considered here. In Figure 4 we show the cross section for fixed $\theta_2 = -\theta_1 = 9°$ (at $\phi_2 = 180°$, the incident and outgoing electron are collinear) for the same energies and atomic number as in the case above. We see that for these angles or momentum transfers, the Bethe-Heitler and Elwert-Bethe-Heitler results overestimate the exact partial wave results.

In our calculations, including $Z = 47$, 53, 60, 68, and 79, with energies E_1 from 50 keV to 450 keV, and k/T_1 from 0.4 to 0.9, we find quite generally that the Bethe-Heitler results overestimate our calculations for small momentum transfers, empirically for $q < 0.6(Z\alpha)\,mc$, and underestimate our calculations for large momentum transfers. For non-forward photon emission angles, and particularly for large photon emission angles, the division into small and large momentum transfers is not as sharp as in the forward emission case, but the results are qualitatively the same. Integration over the outgoing electron angle $d\Omega_2$, effectively integrating over kinematically allowed momentum transfers, "averages out" these differences to some extent, giving the lesser 10-50% differences seen by Fink in $d^2\sigma$. Note we have found that inclusion of the Elwert factor provides mixed results for $d^3\sigma$ – causing greater overestimates in the small momentum transfer range but improving the agreement for large momentum transfers, for the energies considered here. In these energy ranges the Elwert factor improves the integrated result for $d^2\sigma$.

B. Comparison with the Elwert-Haug approximation

In Figure 2 we see that the Elwert-Haug results perform fairly well for small momentum transfers but tend to approach the Elwert-Bethe-Heitler results for large momentum transfers. Generally we find that for $d^3\sigma$ the Elwert-Haug results seldom overestimate the exact partial wave results by more than 20% (although trends in our results indicate that this overestimate will increase with decreasing incident or outgoing electron energy), while for large momentum transfers the Elwert-Haug results tend to seriously underestimate the exact partial wave results. We find that for large momentum transfers or large emission angles the Elwert-Haug results approach the Bethe-Heitler results within a multiplicative constant which is approximately the Elwert-Factor. We find that for the Elwert-Haug results the large momentum transfer region is roughly defined by $q > (Z\alpha)\, mc$. Integration of the Elwert-Haug results over the outgoing electron angle to obtain $d^2\sigma$ typically gives a significant underestimate of the exact partial wave results due to the underestimates at large momentum transfers. This is consistent with Fink's findings that the Elwert-Haug results were as poor as the Bethe-Heitler results for $d^2\sigma$ for high Z elements, for energies similar to those considered here.

IV. SUMMARY AND FUTURE WORK

We have reported a comparison of results obtained using our relativistic partial wave and multipole calculation of the unpolarized triply differential bremsstrahlung cross section with predictions of the (Elwert-)Bethe-Heitler and Elwert-Haug approximations. Our results have been summarized in Table I. For the high Z cases considered here and energies from 50–450 keV we find that there is no significant region of systematic agreement with the (Elwert-)Bethe-Heitler approximation for $d^3\sigma$. At small momentum transfers, $q < 0.6(Z\alpha)\, mc$ for forward emission and with increasing momentum transfers for non-forward emission, the Bethe-Heitler results tend to overestimate the exact partial wave results. For larger momentum transfers they tend to underestimate the exact partial wave results. We see differences ranging from factors of two to factors of four, typically much larger than those seen in the $d^2\sigma$ results. The reduced differences in $d^2\sigma$ are attributed to cancellation of the differences in the small and large momentum transfer regions when integrating over θ_2. This provides an example which confirms the expectation that observation of $d^3\sigma$ provides a more stringent test of the underlying model. In the case of $d^2\sigma$, experimental uncertainty of less than 10-20% is required to differentiate results consistent with Bethe-Heitler approximation, except in the tip region, (or 5-10% if the Elwert factor is included) from those consistent with the exact partial wave calculations. In the case of $d^3\sigma$ measurements can distinguish between the two theories even when experimental uncertainty is as large as 30%. While use of the Elwert factor typically improved agreement in $d^2\sigma$, we find that in $d^3\sigma$ it gives mixed results – causing larger differences at small momentum transfers but improving the agreement somewhat at larger momentum transfers.

In a similar comparison of the Elwert-Haug approximation with the exact partial wave results for the unpolarized cross section for high Z and energies from 50-450 keV, we find that at small momentum transfers the Elwert-Haug results also tend to overestimate the exact partial wave results. However, this overestimation is typically less than 20% for the cases considered here; quite good agreement was found in our calculations for $E_1 = 300$ keV with $E_2 \geq 150$ keV at low momentum transfers and small photon emission angles. We find that for small emission and scattering angles the Elwert-Haug approximation performs well at energies greater than 100 keV. For larger emission angles or larger momentum transfers the Elwert-Haug results tend to significantly underestimate the exact partial wave results, approaching the Elwert-Bethe-Heitler results. We find that the integrated cross section $d^2\sigma$

(as reported by Tseng [16] and Fink [18]) underestimates the exact partial wave results at most energies, reflecting the underestimate at large momentum transfers in $d^3\sigma$.

We now have available results, for coplanar and non-coplanar geometries, over a broad range of atomic species and electron energies, which we can compare with anticipated future experiments [15]. Future theoretical efforts could include examination of bremsstrahlung from electrons of lower or higher energies than considered here. Lower energy cases would be of particular interest in comparisons with predictions of so-called atomic or polarizational bremsstrahlung [19]. At higher energies one should see the nature of convergence toward high energy limit forms. We should also note that limitations in our current computer program have prevented us from considering cases where the photon energy was low compared to the incident electron energy ($k/T_1 < 0.35$). Extension of our code to include such cases could be of interest for comparisons with future experiments.

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FIGURES

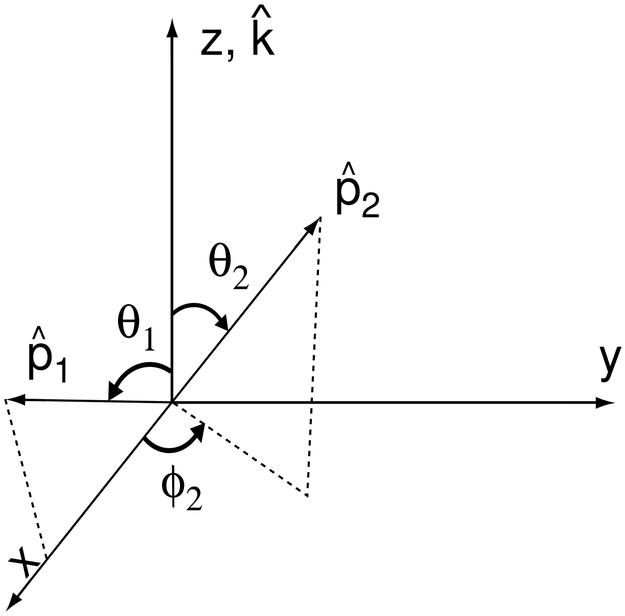

FIG. 1. The coordinate system used in this work. The unit vectors $\hat{p}_1$ and $\hat{p}_2$ are in the direction of the initial and final electron momenta, respectively. The unit vector $\hat{k}$ in the direction of the photon momentum defines the z axis. θ_1 is the angle between $\vec{k}$ and the initial electron momentum $\vec{p}_1$, θ_2 is the angle between $\vec{k}$ and the final electron momentum $\vec{p}_2$, and ϕ_2 is the angle between the plane containing $\vec{k}$ and $\vec{p}_1$ and the plane containing $\vec{k}$ and $\vec{p}_2$.

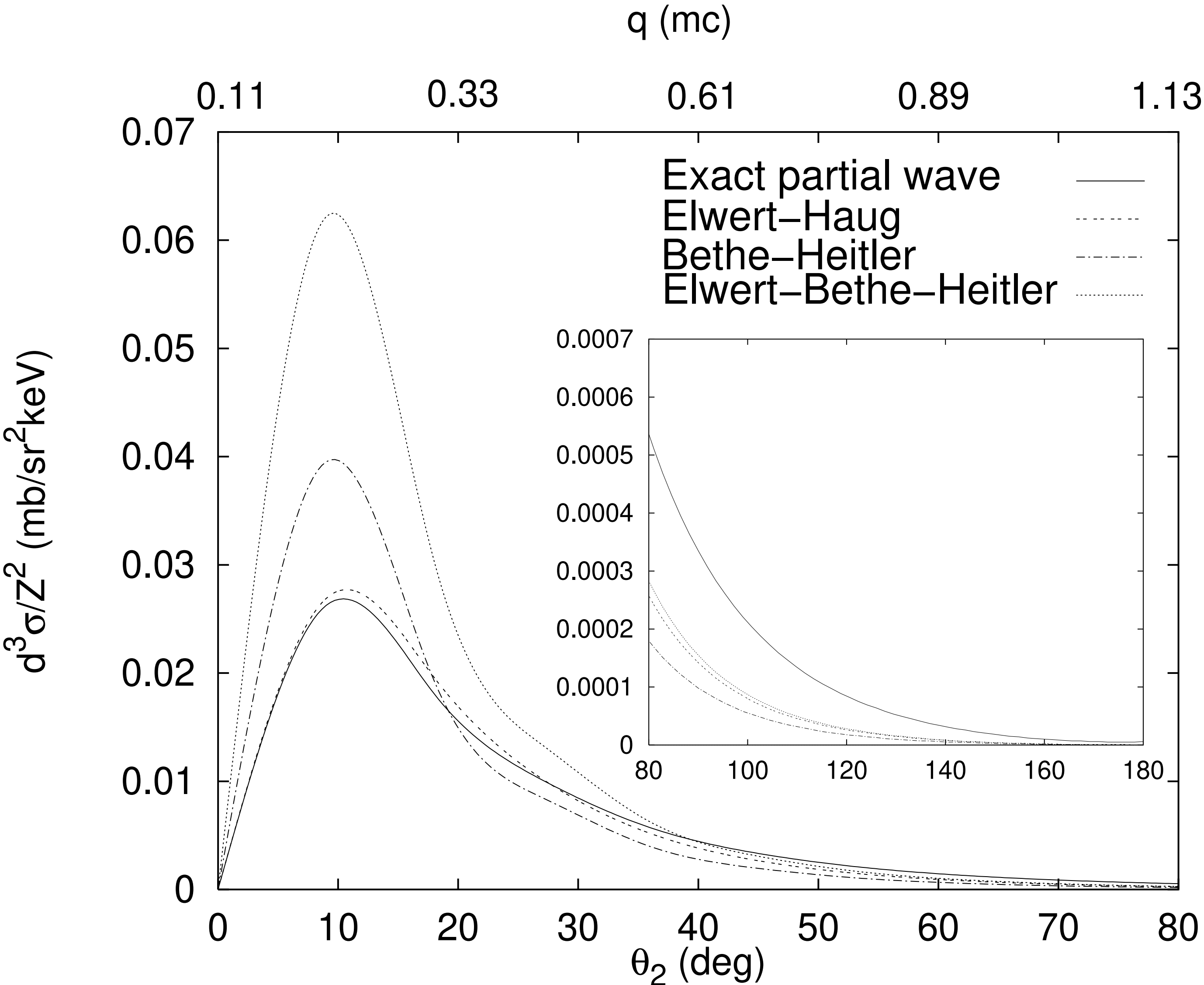

FIG. 2. Forward photon emission $\theta_1 = 0^\circ$ cross sections from the various theories as a function of outgoing electron angle θ_2. The relevant parameters are $Z = 79$, $E_1 = 300$ keV, and $E_2 = 150$ keV. The data has been smoothed using a cubic spline.

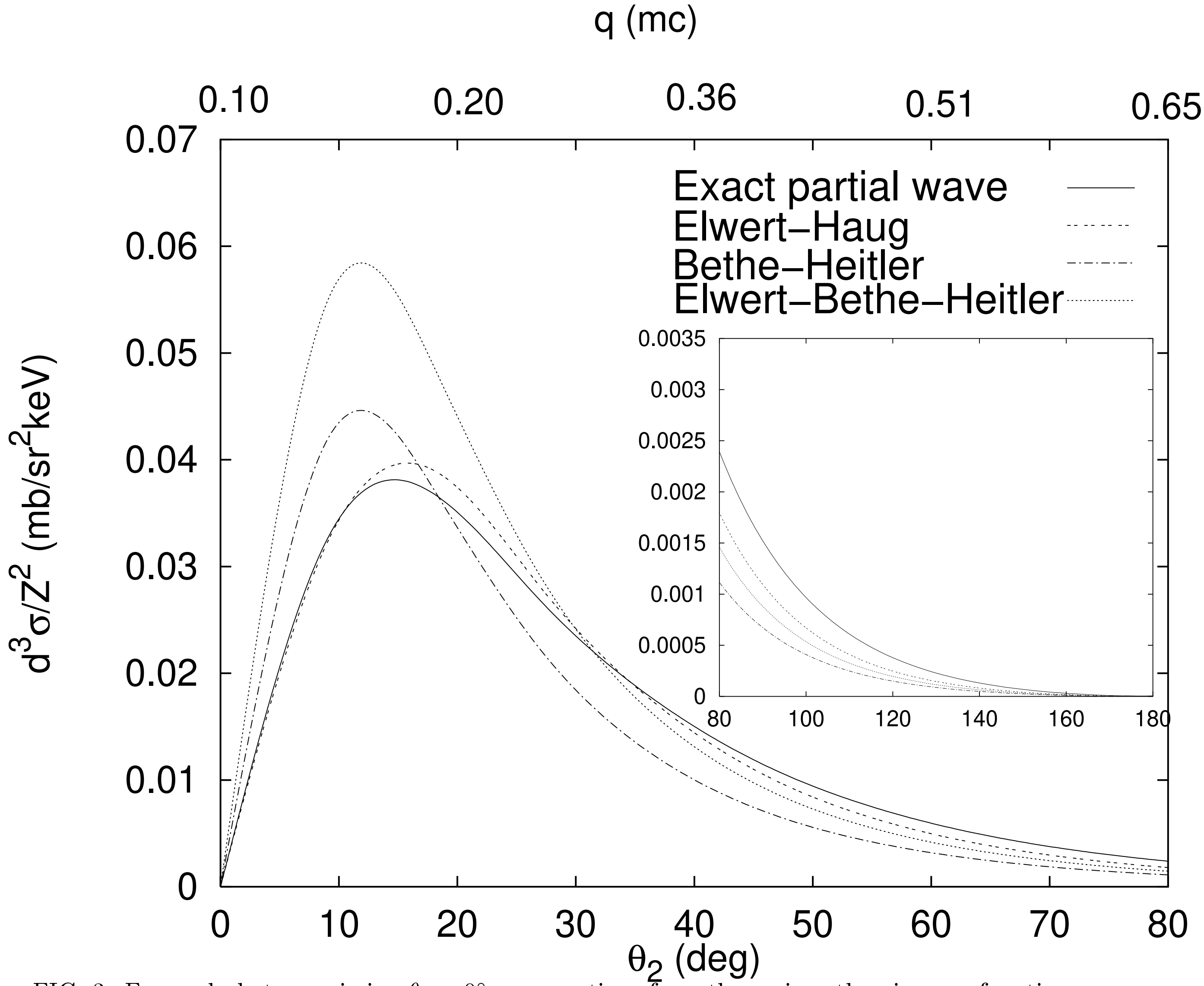

FIG. 3. Forward photon emission $\theta_1 = 0^\circ$ cross sections from the various theories as a function of outgoing electron angle θ_2, for the case $Z = 47$, $E_1 = 100$ keV, and $E_2 = 50$ keV. The data has been smoothed using a cubic spline.

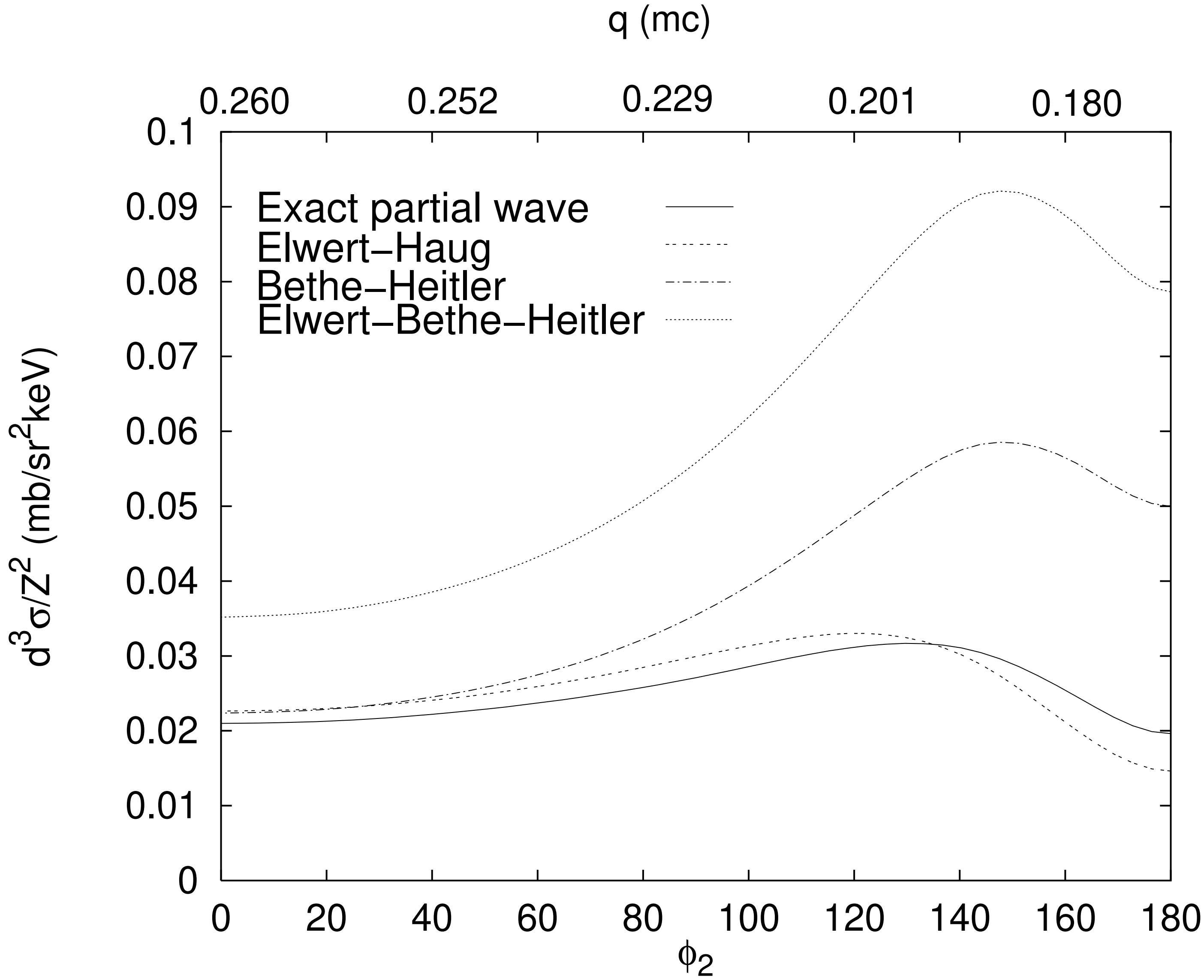

FIG. 4. The cross section as a function of ϕ_2 for $\theta_2 = -\theta_1 = 9°$, $Z = 79$, $E_1 = 300$ keV, $E_2 = 150$ keV. This data has been smoothed using a cubic spline.

TABLES

TABLE I. Ratio of results of approximate theories to exact partial wave theory for $Z \geq 47$, $E_1 = 50$ keV to $E_1 = 450$ keV, with $k/T_1 = 0.4$ to $k/T_1 = 0.85$. Underlined numbers represent dominant cases.

	$d^3\sigma$		$d^2\sigma$
	small q	large q	
Bethe-Heitler [a]	$\underline{2}$ - 4	0.25	0.5 - 0.9
Elwert-Bethe-Heitler	2 - 6	0.3 - 0.5	0.7 - $\underline{1.1}$
Elwert-Haug	1.1 - 1.2	0.3 - 0.5	0.5 - 0.9

[a]These results do not apply in the tip region, where the Bethe-Heitler theory is known to severely underestimate more exact calculations